

# Uncertainty Principle in Loop Quantum Cosmology by Moyal Formalism

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## Abstract

In this paper we derive the uncertainty principle for the Loop Quantum Cosmology homogeneous and isotropic FLWR model with the holonomy-flux algebra. In our derivation we use the Wigner-Moyal-Groenewold phase space formalism. The formalism uses the characteristic functions and the Wigner transform, which maps the quantum operators to the functions on the phase space. The Wigner-Moyal-Groenewold formalism was originally applied to the Heisenberg algebra of the Quantum Mechanics. One can derive from it both the canonical and path integral QM as well as the uncertainty principle. In this paper we apply the phase-space formalism to the quantum cosmology holonomy-flux algebra in case of the homogeneous and isotropic space to obtain the Loop Quantum Cosmology uncertainty principle.

## 1 Introduction

We derive the uncertainty principle for the quantum gravity described by the holonomy-flux algebra in case of the homogeneous and isotropic space. We use the Wigner-Moyal-Groenewold [1] [2] [3] phase space approach. The Wigner transform is the inverse of the Weyl transform from the space of functions to the space of operators. The inverse Fourier transform of the characteristic function becomes a phase-space quasi distribution of the two non-commuting observables. The Moyal's statistical approach can be used to derive the quantum mechanics both in the canonical and path integral forms. It also provides the way to obtain the uncertainty principle by using the cumulants and cumulant generating function. We have noticed that the Wigner-Moyal-Groenewold approach can be applied to derive the uncertainty principle for the Loop Quantum Cosmology in the homogeneous and isotropic case if one uses the holonomy-flux algebra instead of the Heisenberg algebra, i.e.  $[\hat{p}, \hat{N}] = a\mu\hat{N}$  instead of  $[\hat{p}, \hat{q}] = -i\hbar$ , where  $a$  - constant,  $a = \frac{8\pi\gamma G\hbar}{3}$ ,  $\hat{N}$  is an LQC holonomy operator [4], [5]  $\hat{N}_{(\mu)} = e^{i\mu c}$ , where  $c$  - is the configuration variable corresponding to the connection,  $\mu$  - the number of the fiducial cell repetition,  $\mu \in R$ ,  $c \in R_b$  - Bohr real line

compactification. With the assumption that Immirzi parameter  $\gamma$  is real we obtain:

$$\sigma_c^2 \sigma_\mu^2 \geq \left( \frac{4\pi\gamma G}{3} \right)^2 \frac{\hbar^2}{4} \quad (1)$$

The paper is organized as follows. In section 2 we review the Wigner-Moyal-Groenewold phase space approach. In section 3 we derive the holonomy-flux characteristic function. In section 4 we obtain the LQC Wigner operator. In section 5 we obtain the LQC uncertainty principle. The discussion section 6 concludes the paper. Appendix A provides the details of the Moyal's character operator symmetrization in case of the Heisenberg algebra.

## 2 The Wigner-Moyal-Groenewold Phase Space Approach

We will be using Moyal's approach [1] and apply it to the holonomy-flux algebra of the Loop Quantum Cosmology. Therefore we would like first to review the details and derivations of the Moyal's original paper, where the approach was applied to the QM Heisenberg algebra. Most of the details and derivations were omitted in the original Moyal's paper, however they are important for our application that's why we review them in all details.

Given two non-commuting observables  $\hat{p}$  and  $\hat{q}$ , their phase-space mutual quasi distribution is expressed by the inverse Fourier transform of its characteristic function. In order to define the characteristic function and find its Fourier inverse the following operator is introduced:

$$\hat{M}(\tau, \theta) = e^{i(\tau\hat{p} + \theta\hat{q})} \quad (2)$$

For the QM Heisenberg algebra  $[\hat{p}, \hat{q}] = -i\hbar$  it becomes:

$$\hat{M}(\tau, \theta) = e^{i(\tau\hat{p} + \theta\hat{q})} = e^{\frac{1}{2}i\hbar\tau\theta} e^{i\theta\hat{q}} e^{i\tau\hat{p}} = e^{\frac{1}{2}i\tau\hat{p}} e^{i\theta\hat{q}} e^{\frac{1}{2}i\tau\hat{p}} \quad (3)$$

One can easily obtain the above equalities by using Baker-Campbell-Hausdorff formula and the Heisenberg algebra commutation relations (see Appendix A). The original Moyal's paper contains a typo in a sign in this formula. Moyal's version is  $e^{-\frac{1}{2}i\tau\hat{p}} e^{i\theta\hat{q}} e^{\frac{1}{2}i\tau\hat{p}}$ . The typo does not impact the final result or any other derivation in Moyal's paper, however it is very important to correct it before it is used for the holonomy-flux algebra. We use the operator (2) in order to define the characteristic function as a scalar product:

$$M(\tau, \theta) = \langle \psi^*, e^{i(\tau\hat{p} + \theta\hat{q})} \psi \rangle \quad (4)$$

By using the definition above and expanding the two exponents in the second equality of (3) into the Taylor series, we can write:

$$M(\tau, \theta) = \int \psi^*(q) \left(1 + \frac{i\tau\hat{p}}{2} + \dots\right) e^{i\theta\hat{q}} \left(1 + \frac{i\tau\hat{p}}{2} + \dots\right) \psi(q) dq \quad (5)$$

By remembering that  $\hat{q}$  is a multiplication operator, while momentum is a differential operator  $\hat{p} = -i\hbar \frac{d}{dq}$  we can see that:

$$M(\tau, \theta) = \int \psi^*(q) \left(1 + \frac{i\tau}{2}(-i\hbar \frac{d}{dq}) + \dots\right) e^{i\theta \hat{q}} \left(1 + \frac{i\tau}{2}(-i\hbar \frac{d}{dq}) + \dots\right) \psi(q) dq \quad (6)$$

$$M(\tau, \theta) = \int \psi^*(q) \left(1 + \frac{\tau\hbar}{2} \frac{d}{dq} + \dots\right) e^{i\theta \hat{q}} \left(1 + \frac{\tau\hbar}{2} \frac{d}{dq} + \dots\right) \psi(q) dq \quad (7)$$

This is the Taylor expansion of the wave function  $\psi$  and we obtain ([1] formula (3-6))

$$M(\tau, \theta) = \int \psi^*(q - \frac{1}{2}\hbar\tau) e^{i\theta q} \psi(q + \frac{1}{2}\hbar\tau) dq \quad (8)$$

From this formula one can see that  $M(\tau, \theta)$  is a Fourier transform of  $\psi^*(q - \frac{1}{2}\hbar\tau) \psi(q + \frac{1}{2}\hbar\tau)$ .

The phase space distribution function  $F(p, q)$  is an inverse Fourier transform of the characteristic function  $M(\tau, \theta)$  with respect to  $\tau$  and  $\theta$ .

$$F(p, q) = \frac{1}{4\pi^2} \int \int M(\tau, \theta) e^{-i(\tau p + \theta q)} d\tau d\theta \quad (9)$$

,where  $p$  and  $q$  are the eigenvalues of the operators  $\hat{p}$  and  $\hat{q}$ . Moyal assumes that the spectrum is continuous. In case of the discrete spectrum all integrals are replaced by sums.

By substituting (8) into (9) we see that the inverse Fourier transform with respect to  $\theta$  cancels with the forward one and we obtain:

$$F(p, q) = \frac{1}{2\pi} \int \psi^*(q - \frac{1}{2}\hbar\tau) e^{-i\tau p} \psi(q + \frac{1}{2}\hbar\tau) d\tau \quad (10)$$

The characteristic function  $M(\tau|q)$  of  $p$  conditional in  $q$  is

$$M(\tau|q) := \frac{1}{\rho} \int F(p, q) e^{i\tau p} dp \quad (11)$$

,where

$$\rho(q) = \int F(p, q) dp = \psi(q)^* \psi(q) \quad (12)$$

or by substituting (10) and (12) into (11), we obtain:

$$M(\tau|q) = \frac{\psi^*(q - \frac{\tau\hbar}{2}) \psi(q + \frac{\tau\hbar}{2})}{\psi^*(q) \psi(q)} \quad (13)$$

Following Moyal we replace the variables and go to the exponential representation of the wave function:

$$\psi(q) = \rho^{\frac{1}{2}}(q) e^{iS(q)/\hbar} \quad (14)$$

The logarithm of  $M(\tau|q)$  or a cumulant function of  $M(\tau|N)$  is:

$$K(\tau|q) = \log M(\tau|q) = \frac{1}{2} \log \left( \rho(q + \frac{\tau\hbar}{2}) \right) + \frac{1}{2} \log \left( \rho(q - \frac{\tau\hbar}{2}) \right) - \log(\rho(q)) \\ + \frac{i}{\hbar} \left[ S(q + \frac{\tau\hbar}{2}) - S(q - \frac{\tau\hbar}{2}) \right] \quad (15)$$

The cumulants (coefficients of  $\frac{(i\tau)^n}{n!}$  in the Taylor expansion of  $K(\tau|q)$ ) are:

$$\bar{k}_{2n+1}(q) = \left( \frac{\hbar}{2i} \right)^{2n} \left( \frac{\partial}{\partial q} \right)^{2n+1} S(q), \quad \bar{k}_{2n}(q) = \left( \frac{\hbar}{2i} \right)^{2n} \left( \frac{\partial}{\partial q} \right)^{2n} \log \rho(q) \quad (16)$$

,where  $n = 0, 1, \dots$

particularly  $\bar{k}_1(q)$  and  $\bar{k}_2(q)$  are:

$$\bar{k}_1(q) = \frac{\partial S(q)}{\partial q}, \quad \bar{k}_2(q) = \sigma_{p|q}^2 = -\frac{\hbar^2}{4} \frac{\partial^2 \log \rho(q)}{\partial q^2} \quad (17)$$

Moyal's derivation of the Heisenberg Uncertainty Principle by using the characteristic function and the cumulants is as follows ([1] Appendix 1):

$$\bar{k}_1(q) = \bar{p}, \quad \bar{k}_2(q) = \sigma_{p|q}^2 = \overline{p^2} - (\bar{p})^2 \quad (18)$$

where by definition the first and second momentums are:

$$\bar{p} = \frac{1}{\rho} \int F(p, q) dp, \quad \overline{p^2} = \frac{1}{\rho} \int p^2 F(p, q) dp \quad (19)$$

For the two random variables  $\alpha$  and  $\beta$  with zero means we write the Cauchy-Schwarz-Bunyakovsky inequality:

$$|(\overline{\alpha^2} \overline{\beta^2})| = \sigma_\alpha \sigma_\beta \geq |\overline{\alpha\beta}| \quad (20)$$

Taking  $\alpha = \bar{p}$  and  $\beta = q$  and assuming  $\bar{p} = \bar{q} = 0$ , we obtain from (20) :

$$\sigma_q \sigma(\bar{p}) \geq |\overline{q\bar{p}}| = \left| \int q \bar{p} \rho(q) dq \right| \quad (21)$$

Since by definition:

$$\bar{p} = \frac{1}{\rho} \int p F(p, q) dp \quad (22)$$

and

$$\int q \bar{p} \rho(q) dq = \int q p F(p, q) dp dq = \overline{pq} \quad (23)$$

Now reassigning  $\alpha$  and taking:

$$\alpha = \frac{\partial \log \rho}{\partial q}, \quad \bar{\alpha} = \int \frac{\partial \log \rho}{\partial q} \rho dq = \int \frac{\partial \rho}{\partial q} dq = 0 \quad (24)$$

If we differentiate the last equality above with respect to  $q$  we obtain:

$$\frac{\partial}{\partial q} \bar{\alpha} = \frac{\partial}{\partial q} \int \frac{\partial \log \rho}{\partial q} \rho \, dq = 0 \quad (25)$$

We get:

$$\int \left( \frac{\partial \log \rho}{\partial q} \right)^2 \rho \, dq + \int \frac{\partial \log \rho}{\partial q} \frac{\partial \rho}{\partial q} \, dq = \int \left( \frac{\partial \log \rho}{\partial q} \right)^2 \rho \, dq + \int \frac{\partial^2 \log \rho}{\partial q^2} \rho \, dq = 0 \quad (26)$$

from which follows:

$$\bar{\alpha^2} = \int \left( \frac{\partial \log \rho}{\partial q} \right)^2 \rho \, dq = - \int \frac{\partial^2 \log \rho}{\partial q^2} \rho \, dq \quad (27)$$

By using (17) we can express  $\frac{\partial^2 \log \rho}{\partial q^2}$  in terms of  $\sigma_{p|q}$ :

$$\frac{\partial^2 \log \rho}{\partial q^2} = -\frac{4}{\hbar^2} \sigma_{p|q}^2 \quad (28)$$

By substituting it into (27), we obtain:

$$\bar{\alpha^2} = \frac{4}{\hbar^2} \int \sigma_{p|q}^2 \rho \, dq \quad (29)$$

We also note that

$$\overline{\alpha q} = \int q \frac{\partial \log \rho}{\partial q} \rho \, dq = \int q \, d\rho = - \int \rho \, dq = -1 \quad (30)$$

since from (24)  $\alpha = \frac{\partial \log \rho}{\partial q}$

by using the Cauchy-Schwarz-Bunyakovsky inequality (21) where  $\beta = q$  and using (30)

$$|\overline{\sigma_q^2 \alpha^2}| \geq |\overline{\alpha q}| = 1 \quad (31)$$

and multiplying (29) by  $\sigma_q$

$$\overline{\sigma_q^2 \alpha^2} = \sigma_q^2 \frac{4}{\hbar^2} \int \sigma_{p|q}^2 \rho \, dq \geq 1 \quad (32)$$

or

$$\sigma_q^2 \int \sigma_{p|q}^2 \rho \, dq \geq \frac{\hbar^2}{4} \quad (33)$$

By noticing that

$$\sigma_p^2 = \int \left( \sigma_{p|q}^2 + (\bar{p})^2 \right) \rho \, dq \quad (34)$$

and multiplying (34) by  $\sigma_q^2$  from the left and using (21) we obtain:

$$\begin{aligned} \sigma_q^2 \sigma_p^2 &= \sigma_q^2 \int \sigma_{p|q}^2 \rho \, dq + \sigma_q^2 \int (\bar{p})^2 \rho \, dq = \\ &\sigma_q^2 \int \sigma_{p|q}^2 \rho \, dq + \sigma_q^2 \sigma^2(\bar{p}) \geq \sigma_q^2 \int \sigma_{p|q}^2 \rho \, dq + (\overline{qp})^2 \end{aligned} \quad (35)$$

,where in the last equality we used the definition of the  $\sigma^2(\bar{p}) = \int (\bar{p})^2 \rho \, dq$ , while in the last inequality we used the Cauchy-Schwarz-Bunyakovsky inequality (21). By dropping the last term we can rewrite the inequality as:

$$\sigma_q^2 \sigma_p^2 \geq \sigma_q^2 \int \sigma_{p|q}^2 \rho \, dq \quad (36)$$

Finally by using (33) we obtain:

$$\sigma_q^2 \sigma_p^2 \geq \frac{\hbar^2}{4} \quad (37)$$

This completes Moyal's uncertainty principle derivation for the Quantum Mechanics.

### 3 Holonomy-Flux Characteristic Function

Before we can apply Moyal formalism described in the previous section to the Loop Quantum Cosmology holonomy-flux algebra, we need to derive the LQC version of the Wigner's characteristic function. We need to prove that the function we derive has two important properties. First, when integrated by one variable it becomes the distribution density of the other variable. Second, that all momentum integrals used in Moyal derivation still exists when integration is performed with respect to the holonomy-flux measures. The Wigner function in the dual space variables was derived in [6], however in order to apply the Moyal formalism we need to find the Wigner function in the original variables and prove the two mentioned properties.

The holonomy-flux algebra in case of the homogeneous isotropic space FLWR model is [4] [5] [11]

$$[\hat{N}_{(\mu)}, \hat{p}] = -\frac{8\pi\gamma G\hbar}{3} \mu \hat{N}_{(\mu)} \quad (38)$$

The holonomy and flux operators act as follows:

$$\hat{N}_{(\mu)} \Psi(c) = e^{i\mu c} \Psi(c), \quad \hat{p} \Psi(c) = -i \frac{8\pi\gamma G\hbar}{3} \frac{d\Psi}{dc} \quad (39)$$

The basis of the physical Hilbert space is given by LQC analogs of LQG spin-networks:  $\hat{N}_{(\mu)} = e^{i\mu c}$ , where  $c$  - is the configuration variable corresponding to the connection,  $\mu$  - the number of the fiducial cell repetition,  $c \in R_b$  - Bohr compactified real line,  $\mu \in R$ .

The basis functions satisfy the relation:

$$\langle N_{(\mu)} | N_{(\mu')} \rangle = \langle e^{i\mu c} e^{i\mu' c} \rangle = \delta_{\mu, \mu'} \quad (40)$$

The FLWR holonomy-flux algebra commutator is of the form:

$$[\hat{p}, \hat{N}] = a\mu \hat{N} \quad (41)$$

,where  $a$  is a constant:

$$a = \frac{4\pi\gamma G\hbar}{3} \quad (42)$$

Now we are going to obtain the LQC characteristic function  $M(\tau, \theta)$  similar to QM characteristic function (8) and its inverse Fourier transform, which is the Wigner function similar to (9).

Let us formally define the LQC Winger function as:

$$F(\mu, c) = \int \psi^*(c - a\tau) e^{-2ia\tau\mu} \psi(c + a\tau) d\tau \quad (43)$$

where  $\psi(c)$  are the cylindrical functions of the  $c \in R_b$ , compactified real line. The cylindrical functions are of the form:

$$\psi(c) = \sum_{n=0}^N \hat{\Psi}_{\mu_n} e^{i\mu_n c}, \quad \mu_n \in R \quad (44)$$

In order for  $F(\mu, c)$  to have the meaning of the mutual quasi distribution function of  $\mu$  and  $c$  the following two equalities should be true. When integrating with respect to one variable it becomes the distribution density of the other one:

$$\rho_c = \int F(\mu, c) d\mu = |\psi(c)|^2 \quad (45)$$

and

$$\rho_\mu = \int F(\mu, c) dc = |\hat{\Psi}_\mu|^2 \quad (46)$$

In order to prove both equalities we use measures  $dc$  and  $d\mu$  as in [6] [9] [10].

$$\int_{\hat{R}_b} \hat{f}_\mu d\mu = \sum_{\mu \in R} \hat{f}_\mu \quad (47)$$

$$\int_{R_b} e^{i\mu c} dc = \delta_{\mu,0} \quad (48)$$

,where  $\hat{R}_b$  - is Bohr's dual space,  $\delta_{\mu,0}$  - a Kronecker delta.

The characters of the compactified line  $R_b$  are the functions  $h_\mu(c) = e^{i\mu c}$  [9]. The Fourier transform of the function on  $R_b$  is given by :

$$\hat{f}_\mu = \int f(c) h_{-\mu}(c) dc \quad (49)$$

This is an isomorphism of  $L^2(R_b, c) \rightarrow L^2(\hat{R}_b, d\mu)$ .  $e^{i\mu c}$  comprise the basis of  $H = L^2(R_b, dc)$ . Let us prove the equalities (45) and (46). We begin with the first one. We substitute the expression (43) of  $F(\mu, c)$  and the expression (44) for  $\psi(c)$  into (45).

$$\int F(\mu, c) d\mu = \int \int \sum_{n=0}^N \sum_{k=0}^K \hat{\Psi}_{\mu_n}^* e^{-ia\mu_n c} e^{ia\mu_n \tau} \hat{\Psi}_{\mu_k} e^{ia\mu_k c} e^{ia\mu_k \tau} e^{-2ia\tau\mu} d\tau d\mu \quad (50)$$

,where  $\tau \in R_b$ ,  $\mu \in R$ . The integration with respect to  $\mu$  is just a sum as  $\mu$  is discrete.

$$\int F(\mu, c) d\mu = \sum_{\mu \in R} \sum_{n=0}^N \sum_{k=0}^K \int \hat{\Psi}_{\mu_n}^* e^{-ia\mu_n c} e^{ia\mu_n \tau} \hat{\Psi}_{\mu_k} e^{ia\mu_k c} e^{ia\mu_k \tau} e^{-2ia\tau \mu} d\tau \quad (51)$$

By collecting the terms containing  $\tau$  and integrating with respect to  $\tau$ , by using (48) we obtain:

$$\int e^{ia\mu_n \tau} e^{ia\mu_k \tau} e^{-2ia\tau \mu} d\tau = \delta_{2\mu, \mu_k + \mu_n} \quad (52)$$

Since  $\mu \in R$ , summation by  $\mu$  makes the terms with  $2\mu \neq \mu_k + \mu_n$  equal zero and the terms with  $\mu = \mu_k + \mu_n$  equal one and all terms with  $\tau$  and  $\mu$  disappear from the sum. In other words for each pair  $\mu_n$  and  $\mu_k$  there exists  $\mu$  such that  $2\mu = \mu_k + \mu_n$  and that  $\mu$  keeps the  $\mu_k$  and  $\mu_n$  in the sum, all other terms with  $\tau$  zero out in the integration and we obtain:

$$\int F(\mu, c) d\mu = \sum_{n=0}^N \sum_{k=0}^K \hat{\Psi}_{\mu_n}^* e^{-ia\mu_n c} \hat{\Psi}_{\mu_k} e^{ia\mu_k c} = \psi^*(c) \psi(c) = |\psi(c)|^2 \quad (53)$$

In order to prove the equality (46), we substitute the expression (43) of  $F(\mu, c)$  and the expression (44) for  $\psi(c)$  into (46).

$$\int F(\mu, c) dc = \int \int \sum_{n=0}^N \sum_{k=0}^K \hat{\Psi}_{\mu_n}^* e^{-ia\mu_n c} e^{ia\mu_n \tau} \hat{\Psi}_{\mu_k} e^{ia\mu_k c} e^{ia\mu_k \tau} e^{-2ia\tau \mu} d\tau dc \quad (54)$$

,where  $c, \tau \in R_b$ ,  $\mu \in R$

The integration with measure  $dc$  by (48) gives:

$$\int e^{-ia\mu_n c} e^{ia\mu_k c} dc = \delta_{\mu_k - \mu_n, 0} \quad (55)$$

Therefore only the terms with  $\mu_k = \mu_n$  remain in the sums in (54). The integration with respect to  $d\tau$  in turn gives

$$\int e^{ia\mu_n \tau} e^{ia\mu_k \tau} e^{-2ia\tau \mu} d\tau = \delta_{2\mu, \mu_k + \mu_n} \quad (56)$$

From (55) and (56) it follows that:

$$\mu_n = \mu_k = \mu \quad (57)$$

after substituting it into (54) we obtain:

$$\int F(\mu, c) dc = \int \int \hat{\Psi}_{\mu}^* e^{-ia\mu c} e^{ia\mu \tau} \hat{\Psi}_{\mu} e^{ia\mu c} e^{ia\mu \tau} e^{-2ia\tau \mu} d\tau dc \quad (58)$$

the integrals with respect to  $d\tau$  and  $dc$  are equal to one according to (48), so (58) becomes:

$$\int F(\mu, c) dc = \hat{\Psi}_{\mu}^* \hat{\Psi}_{\mu} = |\hat{\Psi}_{\mu}|^2 \quad (59)$$



Thus (53) and (59) imply that  $F(\mu, c)$  is an LQC Wigner function in  $(c, \mu)$  variables. In order to use for the derivation of the LQC uncertainty principle we would need to prove one more equality - the first momentum:

$$\int F(\mu, c) e^{2ia\tau_0\mu} d\mu = \psi^*(c - a\tau_0) \psi(c + a\tau_0) \quad (60)$$

,where  $c, \tau_0 \in R_b$

We begin by substituting the expression of  $F(\mu, c)$  (43) into the l.h.s. of (60)

$$\int F(\mu, c) e^{2ia\tau_0\mu} d\mu = \int \int \psi^*(c - a\tau) e^{-2ia\tau\mu} e^{2ia\tau_0\mu} \psi(c + a\tau) d\tau d\mu \quad (61)$$

By using the expression (44) for the  $\psi(c)$  function, we obtain:

$$\int F(\mu, c) e^{2ia\tau_0\mu} d\mu = \int \int \sum_{n=0}^N \sum_{k=0}^K \hat{\Psi}_{\mu_n}^* e^{-ia\mu_n c} e^{ia\mu_n \tau} \hat{\Psi}_{\mu_k} e^{ia\mu_k c} e^{ia\mu_k \tau} e^{-2ia\tau\mu} e^{2ia\tau_0\mu} d\tau d\mu \quad (62)$$

again, the integration by  $\mu$  can be replaced with the sum over  $\mu$

$$\int F(\mu, c) e^{2ia\tau_0\mu} d\mu = \sum_{\mu \in R} \sum_{n=0}^N \sum_{k=0}^K \int \hat{\Psi}_{\mu_n}^* e^{-ia\mu_n c} e^{ia\mu_n \tau} \hat{\Psi}_{\mu_k} e^{ia\mu_k c} e^{ia\mu_k \tau} e^{-2ia\tau\mu} e^{2ia\tau_0\mu} d\tau \quad (63)$$

The integration by  $\tau$  gives us as before:

$$\int e^{ia\mu_n \tau} e^{ia\mu_k \tau} e^{-2ia\tau\mu} d\tau = \delta_{2\mu, \mu_k + \mu_n} \quad (64)$$

Which means that only those  $\mu$  satisfying  $2\mu = \mu_k + \mu_n$  are equal to one after the integration, all the rest are zeros and we obtain:

$$\int F(\mu, c) e^{2ia\tau_0\mu} d\mu = \sum_n \sum_k \hat{\Psi}_{\mu_n}^* e^{-ia\mu_n c} \hat{\Psi}_{\mu_k} e^{ia\mu_k c} e^{2ia\tau_0\mu} \quad (65)$$

We substitute  $2\mu = \mu_k + \mu_n$  into (65) and use the definition of the LQC cylindrical functions (44):

$$\int F(\mu, c) e^{2ia\tau_0\mu} d\mu = \sum_n \sum_k \hat{\Psi}_{\mu_n}^* e^{-ia\mu_n c} \hat{\Psi}_{\mu_k} e^{ia\mu_k c} e^{ia\tau_0(\mu_n + \mu_k)} = \psi^*(c - a\tau_0) \psi(c + a\tau_0) \quad (66)$$

or by taking  $\tau_0/2$  instead of  $\tau_0$  it can be rewritten in the form:

$$\int F(\mu, c) e^{ia\tau_0\mu} d\mu = \sum_n \sum_k \hat{\Psi}_{\mu_n}^* e^{-ia\mu_n c} \hat{\Psi}_{\mu_k} e^{ia\mu_k c} e^{\frac{ia\tau_0(\mu_n + \mu_k)}{2}} = \psi^*\left(c - \frac{a\tau_0}{2}\right) \psi\left(c + \frac{a\tau_0}{2}\right) \quad (67)$$

This completes the proof of the equality (60).

## 4 LQC Wigner Operator

In this section we would like to find the LQC Wigner operator, i.e the operator with the following property:

$$M(\tau, \theta) = \langle \psi^*, \hat{M} \psi \rangle \quad (68)$$

,where  $\psi$  are the LQC cylindrical functions (44),  $M(\tau, \theta)$  is the LQC characteristic function, which is the Fourier transform of the the quasi probability density function  $F(\mu, c)$  (67) with respect to the group characters:

$$M(\tau, \theta) = \int \int F(\mu, c) e^{i\tau\mu} e^{i\theta c} d\mu dc \quad (69)$$

By substituting  $F(\mu, c)$  expression from (43) into (69) we obtain:

$$M(\tau, \theta) = \int \int \int \psi^*(c - a\tau_0) e^{-2ia\tau_0\mu} \psi(c + a\tau_0) d\tau_0 e^{i\tau\mu} e^{i\theta c} d\mu dc \quad (70)$$

after using the  $\psi(c)$  function definition (44) it becomes:

$$M(\tau, \theta) = \int \int \int \sum_{n=0}^N \sum_{k=0}^K \hat{\Psi}_{\mu_n}^* e^{-i\mu_n c} e^{ia\mu_n \tau_0} \hat{\Psi}_{\mu_k} e^{i\mu_k c} e^{ia\mu_k \tau_0} e^{-2ia\tau_0\mu} d\tau_0 e^{i\tau\mu} e^{i\theta c} d\mu dc \quad (71)$$

As in the previous section the integration with regard to  $\tau_0$  provides the condition by which all terms in the sum zero out except those  $\mu_n$  and  $\mu_k$  satisfying:

$$2\mu = \mu_k + \mu_n \quad (72)$$

The integration over  $\tau_0$  and  $\mu$  provides one as a result. After substituting instead of  $\mu$  its expression from (72) into (71) we obtain:

$$M(\tau, \theta) = \int \sum_n \sum_k \hat{\Psi}_{\mu_n}^* e^{-i\mu_n c} \hat{\Psi}_{\mu_k} e^{i\mu_k c} e^{i\tau \frac{(\mu_k + \mu_n)}{2}} e^{i\theta c} dc \quad (73)$$

Finally we can combine the terms in the exponents and by using the LQC cylindrical function definition (44), write the formula as  $\psi(c - \frac{\tau}{2})$  and  $\psi(c + \frac{\tau}{2})$ :

$$M(\tau, \theta) = \int \psi^*(c - \frac{\tau}{2}) e^{i\theta c} \psi(c + \frac{\tau}{2}) dc \quad (74)$$

Thus we have found the LQC characteristic function  $M(\tau, \theta)$  as a Fourier transform of  $F(\mu, c)$ . Now let us prove that the operator

$$\hat{M}(\tau, \theta) = e^{i\frac{\tau}{a}\hat{p}} \hat{h}_\theta e^{i\frac{\tau}{a}\hat{p}} = e^{i\frac{\tau}{a}\hat{p}} e^{i\theta c} e^{i\frac{\tau}{a}\hat{p}}, \text{ where } \hat{p} = -ia\frac{d}{dc} \quad (75)$$

is an LQC Wigner operator, i.e it satisfies the property (68) for the found LQC characteristic function (74). We can prove (68) directly by substituting the operator (75) into (68)

$$\langle \psi^*, \hat{M} \psi \rangle = \int \psi^*(c) e^{i\frac{\tau}{a}\hat{p}} e^{i\theta c} e^{i\frac{\tau}{a}\hat{p}} \psi(c) dc \quad (76)$$

and expanding the exponents into the Taylor series. We obtain:

$$\begin{aligned}\langle \psi^*, \hat{M} \psi \rangle &= \int \psi^*(c) \left(1 + \frac{i\tau}{2a} \left(-ia \frac{d}{dc} + \dots\right) e^{i\theta c} \left(1 + \frac{i\tau}{2a} \left(-ia \frac{d}{dc} + \dots\right) \psi(c)\right) dc \\ &= \int \psi^*\left(c - \frac{\tau}{2}\right) e^{i\theta c} \psi\left(c + \frac{\tau}{2}\right) dc = M(\tau, \theta)\end{aligned}\quad (77)$$

this is exactly what we aimed to prove:

$$\langle \psi^*, \hat{M} \psi \rangle = M(\tau, \theta) \quad (78)$$

Thus  $\hat{M}(\tau, \theta) = e^{i\frac{\tau}{a}\hat{p}} \hat{h}_\theta e^{i\frac{\tau}{a}\hat{p}} = e^{i\frac{\tau}{a}\hat{p}} e^{i\theta c} e^{i\frac{\tau}{a}\hat{p}}$  is an LQC Wigner operator.

## 5 Loop Quantum Cosmology Uncertainty Principle

We now have all necessary tools ready in order to derive the LQC uncertainty principle similar to Moyal's derivation for the Heisenberg algebra shown in the section 2.

The distribution  $\rho(c)$  was obtained in (53):

$$\rho(c) = \int F(\mu, c) d\mu = \psi(c)\psi^*(c) \quad (79)$$

We define the characteristic function  $M(\tau|c)$  of  $\tau$  conditional in  $c$ .

$$M(\tau|c) = \frac{1}{\rho} \int F(\mu, c) e^{i\tau\mu} d\mu \quad (80)$$

By substituting (67) and (79) into (80) we obtain:

$$M(\tau|c) = \frac{1}{\rho} \int F(\mu, c) e^{i\tau\mu} d\mu = \frac{\psi^*\left(c - \frac{a\tau}{2}\right) \psi\left(c + \frac{a\tau}{2}\right)}{\psi^*(c)\psi(c)} \quad (81)$$

Following Moyal formalism we also replace the variables with the new ones - the amplitude and the phase:

$$\psi(c) = \rho(c)^{\frac{1}{2}} e^{iS(c)/\hbar} \quad (82)$$

The cumulant function of  $M(\tau|c)$  is:

$$\begin{aligned}K(\tau|c) = \log M(\tau, c) &= \frac{1}{2} \log \rho\left(c + \frac{a\tau}{2}\right) + \frac{1}{2} \log \rho\left(c - \frac{a\tau}{2}\right) + \\ &\quad - \log(\rho(c)) + \frac{i}{\hbar} \left[ S\left(c + \frac{a\tau}{2}\right) - S\left(c - \frac{a\tau}{2}\right) \right]\end{aligned}\quad (83)$$

The cumulants (coefficients of  $\frac{(i\tau)^n}{n!}$  in the Taylor expansion of  $K(\tau|c)$ ) are:

$$\bar{k}_1(c) = \frac{1}{i} \frac{\partial K(\tau|c)}{\partial \tau} \Big|_{\tau=0} = \frac{a}{2\hbar} \frac{\partial S(c)}{\partial c} - \frac{-a}{2\hbar} \frac{\partial S(c)}{\partial c} = \frac{a}{\hbar} \frac{\partial S(c)}{\partial c} \quad (84)$$

$$\bar{k}_2(c) = \sigma_{p|c}^2 = \frac{1}{i^2} \frac{\partial^2 K(\tau|c)}{\partial^2 \tau} \Big|_{\tau=0} = -\frac{a^2}{4} \left( \frac{\partial^2 \log \rho(c)}{\partial c^2} \right) \quad (85)$$

As before:

$$\bar{k}_1(c) = \bar{\mu}, \quad \bar{k}_2(c) = \sigma_{\mu|c}^2 = \overline{\mu^2} - (\bar{\mu})^2 \quad (86)$$

The further derivation is similar to Moyal [1], however instead of Heisenberg algebra cumulants we use the holonomy-flux algebra cumulants  $\bar{k}_1(c)$  and  $\bar{k}_2(c)$  to obtain the LQC uncertainty principle.

For the two random variables  $\alpha$  and  $\beta$  with zero means we write the Cauchy-Schwarz-Bunyakovsky inequality:

$$|(\overline{\alpha^2} \overline{\beta^2})| = \sigma_\alpha \sigma_\beta \geq |\overline{\alpha\beta}| \quad (87)$$

Taking  $\alpha = \bar{\mu}$  and  $\beta = c$  and assuming  $\bar{\mu} = \bar{c} = 0$ , we obtain from (87) :

$$\sigma_c \sigma(\bar{\mu}) \geq \left| \int \bar{c} \bar{\mu} \rho(c) dc \right| = |\bar{c} \bar{\mu}| \quad (88)$$

Now taking:

$$\alpha = \frac{\partial \log \rho}{\partial c}, \quad \bar{\alpha} = \int \frac{\partial \log \rho}{\partial c} \rho dc = 0 \quad (89)$$

Like in the case of the Heisenberg algebra we can write

$$\overline{\alpha^2} = \int \left( \frac{\partial \log \rho}{\partial c} \right)^2 \rho dc = - \int \frac{\partial^2 \log \rho}{\partial c^2} \rho dc \quad (90)$$

By expressing  $\frac{\partial^2 \log \rho}{\partial c^2}$  through  $\sigma_{\mu|c}^2$  from (85) we obtain:

$$- \frac{\partial^2 \log \rho}{\partial c^2} = \frac{4}{a^2} \sigma_{\mu|c}^2 \quad (91)$$

Then by substituting it into (90) we get:

$$\overline{\alpha^2} = - \int \frac{\partial^2 \log \rho}{\partial c^2} \rho dc = \int \frac{4}{a^2} \sigma_{\mu|c}^2 \rho dc \quad (92)$$

and similar to (30)

$$\overline{\alpha c} = \int c \frac{\partial \log \rho}{\partial c} \rho dc = -1 \quad (93)$$

by multiplying (92) by  $\sigma_c^2$  and using the Cauchy-Schwarz-Bunyakovsky inequality (87) where  $\beta = c$  and using

$$|\overline{\sigma_c^2 \alpha^2}| \geq |\overline{\alpha c}| = 1 \quad (94)$$

we obtain:

$$\overline{\sigma_c^2 \alpha^2} = \frac{4\sigma_c^2}{a^2} \int \sigma_{\mu|c}^2 \rho dc \geq 1 \quad (95)$$

by assuming that Immirzi  $\gamma$  is real and therefore  $a^2 > 0$ , we get

$$\sigma_c^2 \int \sigma_{\mu|c}^2 \rho dc \geq \frac{a^2}{4} \quad (96)$$

By noticing that

$$\sigma_\mu^2 = \int \left( \sigma_{\mu|c}^2 + (\bar{\mu})^2 \right) \rho \, dc \quad (97)$$

exactly as in (35), by multiplying (97) by  $\sigma_c^2$  from the left we obtain:

$$\begin{aligned} \sigma_c^2 \sigma_\mu^2 &= \sigma_c^2 \int \sigma_{\mu|c}^2 \rho \, dc + \sigma_c^2 \int (\bar{\mu})^2 \rho \, dc = \\ &\sigma_c^2 \int \sigma_{\mu|c}^2 \rho \, dc + \sigma_c^2 \sigma^2(\bar{\mu}) \geq \sigma_c^2 \int \sigma_{\mu|c}^2 \rho \, dc + (\overline{c\mu})^2 \end{aligned} \quad (98)$$

,where in the last inequality above we used (88). By dropping the last term we can rewrite the inequality as:

$$\sigma_c^2 \sigma_\mu^2 \geq \sigma_c^2 \int \sigma_{\mu|c}^2 \rho \, dc \quad (99)$$

Finally by using (96) we obtain:

$$\sigma_c^2 \sigma_\mu^2 \geq \frac{a^2}{4} \quad (100)$$

By remembering that expression for  $a = \frac{4\pi\gamma G\hbar}{3}$  we can rewrite it as:

$$\sigma_c^2 \sigma_\mu^2 \geq \left( \frac{4\pi\gamma G}{3} \right)^2 \frac{\hbar^2}{4} \quad (101)$$

## 6 Discussion

By using Moyal's approach for the holonomy-flux algebra, we have obtained the uncertainty principle for the Loop Quantum Cosmology in the case of homogeneous and isotropic space:

$$\sigma_c^2 \sigma_\mu^2 \geq \left( \frac{4\pi\gamma G}{3} \right)^2 \frac{\hbar^2}{4} \quad (102)$$

We have made the assumption that the Immirzi parameter is real. We have obtained the Wigner function on the space of the cylindrical wave functions defined on  $R_b$  and the form of the LQC Wigner operator. The result for the FLWR holonomy-flux algebra symmetries is quite similar to Heisenberg algebra, even though the representation, the Hilbert space, and the integration measures are very different,  $\mu$  is discrete and  $c$  belongs to the compact  $R_b$  space.

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## 7 Appendix A BCH Formula for Heisenberg Algebra

$$\hat{M}(\tau, \theta) = e^{i(\tau \hat{p} + \theta \hat{q})} \quad (103)$$

For the QM Heisenberg algebra  $[\hat{p}, \hat{q}] = -i\hbar$  it becomes:

$$\hat{M}(\tau, \theta) = e^{i(\tau\hat{p}+\theta\hat{q})} = e^{\frac{1}{2}i\hbar\tau\theta} e^{i\theta\hat{q}} e^{i\tau\hat{p}} = e^{\frac{1}{2}i\tau\hat{p}} e^{i\theta\hat{q}} e^{\frac{1}{2}i\tau\hat{p}} \quad (104)$$

To show the first equality we just use the BCH formula for the algebra Heisenberg algebra:

$$e^{i\theta\hat{q}} e^{i\tau\hat{p}} = e^{i(\theta\hat{q}+\tau\hat{p})+\frac{1}{2}[i\theta\hat{q}, i\tau\hat{p}]} = e^{i(\tau\hat{p}+\theta\hat{q})-\frac{\tau\theta i\hbar}{2}} \quad (105)$$

it follows then

$$e^{i(\tau\hat{p}+\theta\hat{q})} = e^{\frac{1}{2}i\tau\theta\hbar} e^{i\theta\hat{q}} e^{i\tau\hat{p}} \quad (106)$$

The second equality in (104) can be shown by using the BCH formula directly in two steps:

$$\begin{aligned} e^{\frac{1}{2}i\tau\hat{p}} e^{i\theta\hat{q}} e^{\frac{1}{2}i\tau\hat{p}} &= e^{\frac{1}{2}i\tau\hat{p}} (e^{i(\frac{\tau}{2}\hat{p}+\theta\hat{q})-\frac{\tau\theta i\hbar}{4}}) = e^{-\frac{1}{4}i\tau\theta\hbar} e^{\frac{1}{2}i\tau\hat{p}} e^{i(\frac{\tau}{2}\hat{p}+\theta\hat{q})} = \\ &= e^{-\frac{1}{4}i\tau\theta\hbar} e^{(\frac{i\tau}{2}\hat{p}+\frac{i\tau}{2}\hat{p}+i\theta\hat{q})} e^{\frac{1}{4}i\tau\theta\hbar} = e^{i(\tau\hat{p}+\theta\hat{q})} \quad (107) \end{aligned}$$

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